The method of virtual work revisited

H. Christiaen

Assistant Professor of Mechanics

Katholieke Universiteit Leuven

Abstract: A virtual displacement is mostly presented as some very small imagininary or hypothetical displacement. An approach to this method is given, which is easier to understand for undergraduates and agrees better with a modern interpretation of differentials. This presentation uses virtual velocities instead of virtual displacements.

Problems with the common presentation

In most textbooks, a virtual displacement is presented as some very small imaginery or hypothetical displacement. When I was first thaught the method of virtual work, it seemed to me that, if you need a small displacement for determining the equilibrium, then there should be some interval of equilibrium. Or it is well known that only some special forces, such as friction, lead to an interval of equilibrium. The method of virtual work has some problems with treating friction and normally leads to a point of equilibrium.

When I was myself teaching, another problem arose. The professor teaching analysis and differential equations did not like people talking about 'infinitesimal' quantities. When a student would say that dx represents an infinitesimal displacement along the x-axis or infinitesimal variation of the x-value, he would ask how small an infinitesimal displacement is? For him dx is the limit of Δx and does no longer represent an interval. So if we present the δx in the method of virtual work as a very small displacement and in the course of analysis dx may not be called a displacement, then we have a conflict in the presentation of concepts between different courses.

In an article in this journal of April 1989 it was said : 'let a compatible virtual displacement of a body be defined as a set of imaginery first-order differential displacements, which …'. This definiton is related to another question which bothered me for some time. Why are analytical solutions of differential equations exact, although nothing is said about higher order derivatives?

Or, in other words, why are differential equations clearly NOT first-order approximations? The answer is that they are exact because they give the relation between variations of variables not in an interval but in a point.[1] This is a bit beyond our imagination, but we commonly use other notions which are beyond our imagination. A geometrical point is also something we can not imagine. When we draw a point, then we create something which is at least two dimensional, and a point made with chalk on a blackboard is clearly three dimensional! Still, no student has problems with the notion of geometrical point. So there is an essential difference between a Δx and a dx. When we write $\Delta y = f'(x)\, \Delta x$, we are talking about an approximation in an interval. When we write $dy = f'(x)\, dx$, this is no longer an approximation but an exact relation, and no longer in an interval but in a point.

It is not the intention of this article to revise the theoretical foundations of the method of virtual work, but only the presentation which could be used for undergraduates.

Virtual work without virtual displacement?

Can we get out of these problems and use the modern treatment of differentials for presenting the method of virtual work? I will try to do that in this article. First we must return to the expression $dy = f'(x)\, dx$. If $dy = 0$ then there is only one explantion: $f'(x) = 0$. Nobody would say that there are 2 possibilities: or $f'(x) = 0$ or $dx = 0$. 'dx' here just indicates the variable involved, but is not an interval which might be 0.

A simple derivation of the method of virtual work starts from the differential form of the energy law for one particle:

$$\sum \vec{F}_i \bullet d\vec{r} = m\vec{a} \bullet d\vec{r} \qquad (1)$$

In statics, we are looking for a position where the accelereation is 0. Then $m\vec{a} = 0$ is equivalent with $\sum \vec{F}_i = 0$ .This means that the left handside of the previous equation (1) equals 0. Thus far nothing new seems to be said. But when we replace $d\vec{r}$ by $\vec{v}dt$, then the differential becomes $(\sum \vec{F}_i \bullet \vec{v})\, dt$ and the condition for equilibrium becomes $\sum \vec{F}_i \bullet \vec{v} = 0$. For students, a velocitiy is far more understandable than a differential. They know very well that the velocity vector has the direction of the tangent to the path. The velocity in this formula may be considered as the veolicty with which the particle might pass through the point of equilibrium. As the left handside of the equation must be zero, the magnitude of this velocity is irrelevant. This formulation is easier to understand than the previous one. Students also know that a dot product will be 0 if both arguments are perpendicular to each other. This is the case in some ideal constraints, where the force is perpendicular to the displacement. Ideal constraints are here defined as constraints which do not involve exchange of energy between the system and its environment.

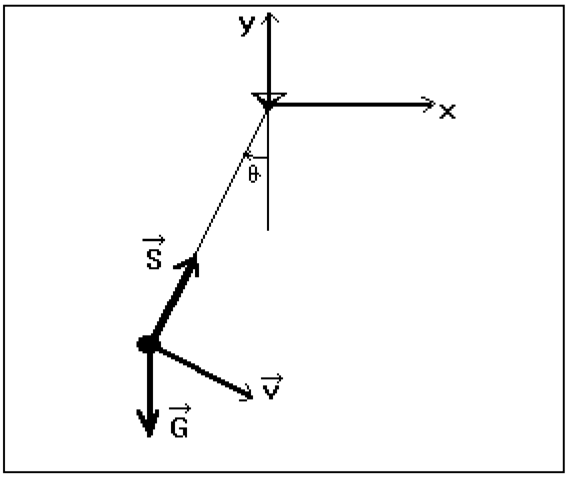

Figure 1 : a very simple system

The dimension of the expression $\sum \vec{F}_i \bullet \vec{v}$ is no longer work, but power. It says that the system can be in equilibrium in a point where the total power transferred into or out of the system is null. The method is most interesting when we have ideal constraints in the system because they may be dropped from the calculations. This power approach now says something different from the formulation $\sum \vec{F}_i = 0$. It gives a condition involving only the active forces, given that the reaction forces in the ideal constraints can be delivered. This condition is sometimes called the strict condition for equilibrium.

The very simple example of the mathematical pendulum may illustrate all this (fig. 1). The force S in the rope is always perpendicular to the velocity: it is an ideal constraint and is not involved in exchange of energy with the system. The condition of equilibrium reduces to $\vec{G}$ being perpendicular to $\vec{v}$, which is clearly the case in the lowest point. In order to find this point, there is no need for any infinitesimal displacement.

More degrees of freedom

For a more realistic problem of virtual work, with more than one degree of freedom and associated generalized co-ordinates q_j, we can write the velocity of point i as a function of the q_j:

$$\vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{dq_j}{dt}$$

This expression gives the virtual velocity as distinct from the real velocity, which takes also into account the explict presence of time dependency in the tranformation equations. This time dependency can only be present in the case of a dynamic equilibrium. I will not dwell here on the question why this explicit time dependency should not be taken into account. Virtual velocities are distinguished from the real velocities in the same way as the virtual displacements are distinct from the real displacements.

These virtual velocities are used by several authors instead of the virtual displacements.[2-3] Some of them call the method then consequently 'method of virtual power' (in French : 'méthode des puissances virtuelles' instead of 'métode du travail virtuel' [2]) others say that it gives the work in a unit of time [3]. The condition for equilibrium now becomes:

$$\sum_i \vec{F}_i \bullet \vec{v}_i = \sum_i \sum_j \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{dq_j}{dt} = 0 \qquad (2)$$

The left handside is a function of several variables. We may group the terms of this sum in function of the $\frac{dq_j}{dt}$, the generalized velocities. As the q_j are independent variables, also these velocities are independent. They represent the velocities with which the points of the system might move along the parametric curves. So in order for this equation to be zero, each coefficient of these velocities will have to be zero. So the condition of equilibrium is now a set of equations:

$$\sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{dq_j}{dt} = 0 \quad \text{for each } q_j \qquad (3)$$

In this formula, each force contributes to the result according to a ceratin weight factor. This weight is composed of:

1. The projection of the force on the tangent to the parmetric curve for q_j passing through $\vec{r}_i$. This is done by the dot product.
2. The ratios of the virtual velocities of the different points involved. As the sum must equal zero, only the ratios matter and the exact magnitude of these velocities is irrelevant. This

means that a force acting on a point moving quickly will have more influence than a same force acting on a point moving slowly.

Back to the classical form

When we look at equation (3), we see that the generalized velocity may be factorized out and may even be dropped from the formula. Calculating this generalized velocity is doing to much work. We can use as condition for equilibrium:

$$\sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_j} = 0 \quad \text{for each } q_j \qquad (4)$$

For this, there is no need to consider some displacement of the system.[4]

But there are certain advantages in using the classical form with differentials:

$$\sum_i \vec{F}_i \bullet \delta\vec{r}_i = \sum_j \sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j = 0 \qquad (5)$$

The left handside of this equation is a differential in several independent variables. In order for this differential to be zero, the function in front of each δq_j must be zero, which gives us again equation (4). This equation written as a differential becomes:

$$\sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j = 0 \quad \text{for each } q_j \qquad (6)$$

The use of the differentials has the advantage that we see clearly what variables we are using. If we have a system with 1 degree of freedom and we end up with an expression with more than 1 differential δq_i , then we have to look for supplementary equations of constraint in order to write the whole expression as a function of 1 variable. To me it seems that the easiest way of solving this kind of problem is by differentiating the transformation equations immediately with respect to all generalized co-ordinates, introduce these in the equation $\sum_i \vec{F}_i \bullet \delta\vec{r}_i = 0$ and group according to the the δq_j .

Conclusion

A virtual displacement is for me a displacement which takes only into account the direct influence of the generalized co-ordinates in the tranformation equations. They are distinguished from the real velocities in the same way as the virtual displacements are distinct from the real displacements. The use of virtual velocities gives a better understanding of the whole mechanism of the method of virtual work, without needing any imaginary or infinitesimal displacement. So I think that this is a better way of presenting this method. But writing down the computations in terms of differentials is probably a more safe method than trying to use directly the equation (4).

References

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